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Vorlesung

Netzalgorithmen

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2.1 Modeling Design Problems in Link-Path Formulation

Preliminary notation for undirected demands:

- $x - y$ ...Link between node x and y
- $\hat{h}_{xy} = b$ ...Demand d between nodes x and y (i. e. Demand b bandwidth between nodes x and y)
- $\hat{x}_{v_1 v_2} = f$ ...Flow on the path between v_1 and v_2
- $\hat{x}_{v_1 v_2 v_3} = f$ ...Flow on the path v_1, v_2, v_3
- $\hat{c}_{v_1 v_2}$ ...Capacity of the link between v_1 and v_2

This generally yields equations for satisfying the demands and inequalities for limiting the bandwidth to the link capacities, e. g.

$$\hat{x}_{12} + \hat{x}_{132} = 5 \tag{2.1}$$
$$\hat{x}_{13} + \hat{x}_{123} = 7 \tag{2.2}$$
$$\vdots \tag{2.3}$$
$$\hat{x}_{132} + \hat{x}_{13} + \hat{x}_{213} \leq 10 \tag{2.4}$$
$$\hat{x}_{132} + \hat{x}_{123} + \hat{x}_{23} \leq 15 \tag{2.5}$$

As such a system usually has multiple solutions, an **objective function** is defined, which can then be optimized, e. g.:

$$F = \hat{x}_{12} + 2\hat{x}_{132} + \hat{x}_{13} + 2\hat{x}_{123} + \hat{x}_{23} + 2\hat{x}_{213} \tag{2.6}$$

In this example, flow routed over two links are weighted with factor two, so direct links are preferred.

Such an optimization is called a **multi-commodity flow problem**. The *optimal solution* to this problem is marked with an asterisk, e. g.:

$$\hat{x}^*_{12} = 5 \qquad \hat{x}^*_{13} = 7 \qquad \hat{x}^*_{23} = 8 \tag{2.7}$$

Important observations:

- Changing the objective function usually affects the optimal solution to a problem.
- Formulation a good objective function for the particular network is important for obtaining meaningful solutions.

2.2 Modeling Design Problems in Node-Link Formulation

In this section, links and demands are assumed to be directed. Undirected links $x - y$ are replaced with two directed links $x \to y$ and $y \to x$.

Notation:

- $v_1 \to v_2$ ...Directed link from node v_1 to node v_2
- $\langle v_1 : v_2 \rangle$ ...Demand from node v_1 to node v_2
- $\tilde{x}_{a,d}$ ...Flow over arc a for demand d (e. g. $\tilde{x}_{v_1 v_2, v_1 v_2}$ flow over arc $v_1 \to v_2$ for demand $\langle v_1 : v_2 \rangle$)

Backflows (e. g. $\tilde{x}_{21,12}$) are also possible, but make practically no sense, so they are set to 0.

2.3 Link-Demand-Path-Identifier-Based Notation

Demands and links are assigned label indexes. Thus, tables for mapping indexes to actual demands and links are required.

Notation:

- h_i ...Demand with index i
- c_e ...(Known) capacity of link e
- y_e ...*Unknown* capacity of link e
- P_i ...Number of candidate paths for demand i
- P_{ij} ...jth candidate path for demand i
- $(v_1, e_1, v_2, e_2, \ldots, e_n, v_{n+1})$ ...n-hop path
- $v_1 \to v_2 \to \ldots \to v_{n+1}$ ...node representation of a path (directed, use $-$ instead of $\to$ for undirected)
- $\{e_1, e_2, \ldots, e_n\}$ ...link representation of undirected paths
- $(e_1, e_2, \ldots, e_n)$ ...link representation of directed paths
- v ...Node
- e ...Link
- d ...Demand
- p ...Path
- V, E, D, P ...total numbers of the aforementioned items

- ξ_e ...Cost of alink e

Example equations:

$$x_{11} = 15 \tag{2.8}$$
$$x_{21} + x_{22} = 20 \tag{2.9}$$
$$x_{31} + x_{32} = 10 \tag{2.10}$$

Demands:

$$\sum_{p=1}^{P_d} x_{dp} = h_d \qquad d = 1, \ldots, D \tag{2.11}$$

Short form, when iterating over all candidate paths:

$$\sum_{p} x_{dp} = h_d \qquad d = 1, \ldots, D \tag{2.12}$$

The vector of all flows (path flow variables) is called the **flow allocation vector** or short **flow vector**:

$$\begin{aligned} \mathbf{x} &= (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_D) &(2.13)\\ &= (x_{11}, x_{12}, \ldots, x_{1P_1}, \ldots, x_{D1}, x_{D2}, \ldots, x_{DP_D}) &(2.14)\\ &= (x_{dp} : d = 1, 2, \ldots, D; p = 1, 2, \ldots, P_d) &(2.15)\end{aligned}$$

Important: vectors are represented with bold letters $\mathbf{x}$ and scalar values are represented with normal letters x.

The relationship between links and paths is written down with the **link-path incidence relation** δ_{edp}:

$$\delta_{edp} = \begin{cases} 1 & \text{if link } e \text{ belongs to path } p \text{ for demand } d \\ 0 & \text{otherwise} \end{cases} \tag{2.16}$$

δ_{edp} can be written as a table:

e	$P_{11} = \{2, 4\}$	$P_{21} = \{5\}$	$P_{22} = \{3, 4\}$
1	0	0	0
2	1	0	0
3	0	0	1
4	1	0	1
5	0	1	0

The **load** in link e can be written as:

$$\underline{y}_e = \underline{y}_e(\mathbf{x}) = \sum_{d=1}^{D} \sum_{p=1}^{P_d} \delta_{edp} x_{dp} \tag{2.17}$$

Important: The actual link loads $\underline{y}_e$ are determined by the path flow variables x_{dp} of a solution and are not the same as the link capacity variables y_e.

Cost of a path P_{dp}:

$$\zeta_{dp} = \sum_e \delta_{edp} \xi_e, \qquad d = 1, 2, \ldots, D \quad p = 1, 2, \ldots, P_d \tag{2.18}$$

Shortest-Path Allocation Rule for Dimensioning Problems For each demand, allocate its entire demand to its shortest path with respect to link costs and candidate paths. If there is more than one shortest path for a given demand, then the demand volume can be arbitrarily split amon the shortest paths.

2.4 Shortest-Path Routing

For shorted-path routing, demands will only be routed on their shortest paths. The path length is determined by adding up link costs w_e according to some weight system $\mathbf{w} = (w_1, w_2, \ldots, w_E)$.

Single Shortest Path Allocation Problem For given link capacities $\mathbf{c}$ and demand volumes $\mathbf{h}$, find a link weight system $\mathbf{w}$ such that the resulting shortest paths are unique and the resulting flow allocation vector is feasible.

Zusammenfassung fortführen

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3.1 Extreme Points and Basic Solutions

$$S(x) \text{ linear unabhängig, wenn } \sum_{i=1}^{k} \alpha_i \vec{x}_i = 0 \implies \forall i \in \{1, \ldots, k\} : \alpha_i = \vec{o} \tag{3.1}$$

Stichwortverzeichnis